

Integrable Generalized Thirring Model

Korkut Bardakçi, Luis M. Bernardo * and Nir Sochen **

Department of Physics[†]

University of California at Berkeley

and

Theoretical Physics group

Lawrence Berkeley Laboratory

Berkeley, CA 94720, U.S.A.

Abstract

We derive the conditions that the coupling constants of the Generalized Thirring Model have to satisfy in order for the model to admit an infinite number of commuting classical conserved quantities. Our treatment uses the bosonized version of the model, with periodic boundary conditions imposed on the space coordinate. Some explicit examples that satisfy these conditions are discussed. We show that, with a different set of boundary conditions, there exist additional conserved quantities, and we find the Poisson Bracket algebra satisfied by them.

June. 96

* Supported by JNICT, Lisbon.

** e-mail: sochen@asterix.lbl.gov

[†] This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-90-21139.

1. Introduction

The purpose of this paper is investigate the conditions under which the Generalized Thirring Model becomes integrable. The Generalized Thirring Model is the field theory of massless fermions in two dimension, interacting through the most general four fermi interaction compatible with Lorentz invariance [1]. By integrability, we mean the existence of an infinite number of conserved and commuting dynamical variables. In this paper, we will not address the question whether these are sufficient in number to make the model truly solvable. The bosonized form of the Thirring model will be our starting point, and our treatment from that point on will be purely classical. It should, however, be noted that this is better than treating the original fermionic model classically, since bosonization does capture some of the quantum nature of the model. From the bosonized Lagrangian, we wish to extract a Lax pair depending on a spectral parameter. For this purpose, we first write the equations of motion in a suggestive form as flatness conditions for two vector fields, and we demand the existence of another flat vector field that interpolates between these two. This results in equations involving the coupling constants which we call the first integrability condition. In general, these equations are overdetermined and they do not have solutions depending on a continuous parameter. In section 3, we discuss four exceptional cases when such a solution exists. The first case is the model with maximum internal symmetry and a single coupling constant. The second case is a simple generalization of the first one to a product group with two different coupling constants. The third example is an $SU(2)$ model broken down to $U(1)$. The last example is a model based on symmetric spaces. For each of these examples, there exists a Lax pair depending on a spectral parameter.

Given the Lax pair, in a standard fashion, one can construct conserved quantities in terms of a path ordered product. It is, however, necessary to specify boundary conditions. The simplest boundary condition is the periodic one, with the space coordinate compactified into a circle. Taking the trace of the path ordered product and expanding in powers of the spectral parameter yields an infinite number of conserved “charges”. In section 4, we compute the Poisson brackets of these charges and derive the conditions so that it vanishes. This is then our second integrability condition. In the four examples discussed earlier, the second integrability condition is automatically satisfied, although we are unable to prove in general that the second condition follows from the first. Another natural boundary condition is the open one: The space interval is from $-\infty$ to $+\infty$ and the fields vanish at $\pm\infty$. In this case, additional integrals of motion can be constructed by considering the

matrix elements of the path ordered product, instead of just the trace. In section 5, we compute the Poisson brackets of these additional integrals of motion and find that they satisfy a non-linear algebra. We also compare this algebra to a simpler algebra derived in a somewhat similar case of the principal chiral model [2]. The last section summarizes our conclusions.

2. Lax Pair Formulation

We begin this section by recalling the definition of the model, the bosonized version of it and its equations of motion and symmetries (see [1] and [3] for detailed analysis). Lightcone variables will be used throughout: x_+ will serve as time and x_- as space.

The parity violating generalized Thirring model is given by the action

$$S_o = \int d^2x (\bar{\Psi} i \gamma^\mu \partial_\mu \Psi - (\tilde{G}^{-1})^{ab} \bar{\Psi}_R \tilde{t}_a \Psi_R \bar{\Psi}_L \tilde{t}_b \Psi_L), \quad (2.1)$$

where R and L refer to the right and left chiral components of Ψ , and \tilde{t}_a are the generators of the Lie algebra \mathcal{G} in some representation:

$$[\tilde{t}_a, \tilde{t}_b] = i f_{ab}^c \tilde{t}_c.$$

We reserve the notation t_a for the adjoint representation. The coupling constant $(\tilde{G}^{-1})^{ab}$ is an invertible not necessarily symmetric matrix, resulting in parity violation. In one version of bosonization [4][1][5], this gives

$$S_o = W(g) + W(h^{-1}) - \frac{n}{2\pi} \int d^2x G_{ab} (i g^{-1} \partial_+ g)^a (i h^{-1} \partial_- h)^b, \quad (2.2)$$

where X_a stands for $Tr(t_a X)$, g and h are group elements expressed in the adjoint representation, and

$$G_{ab} = \frac{1}{2} \kappa_{ab} - \frac{\pi}{2n} \tilde{G}_{ab}, \quad (2.3)$$

where κ_{ab} is the Cartan-Killing metric $\kappa_{ab} = Tr(t_a t_b)$, n is the number of fermion flavors shifted by the dual Coxeter number and W is the WZW action

$$W(g) = \frac{n}{8\pi} \left(\int d^2x Tr(\partial_\mu g^{-1} \partial^\mu g) + \frac{2}{3} \int Tr((g^{-1} dg)^3) \right). \quad (2.4)$$

The equations of motion are equivalent to conservation of two currents:

$$\partial_+ J_- = \partial_- J_+ = 0, \quad (2.5)$$

where

$$\begin{aligned} J_+ &= i \frac{n}{4\pi} \left(-\partial_+ h h^{-1} + h \tilde{t}_a h^{-1} (2G^T)^a_b (g^{-1} \partial_+ g)^b \right), \\ J_- &= i \frac{n}{4\pi} \left(-\partial_- g g^{-1} + g \tilde{t}_a g^{-1} (2G)^a_b (h^{-1} \partial_- h)^b \right). \end{aligned} \quad (2.6)$$

Next we notice that the model is invariant under the transformation $h \rightarrow u_+(x_+)h$, with J_+ transforming like a gauge field. Using this transformation, J_+ , which only depends on x_+ , (see (2.5)), can be set equal to zero. This gives us a special solution to the equations of motion; the general solution is obtained by applying the inverse transformation¹. For this special solution, treating x_+ as time, we will search for an infinite set of conserved quantities, which assure the integrability of the model. The general solution is also integrable by virtue of the transformation introduced above. The situation is similar to what happens in the WZW model [6]. The conserved quantities are derived for this special solution with $J_+ = 0$. Define now

$$V_\pm^a = (i h^{-1} \partial_\pm h)^a, \quad (2.7)$$

$$W_+^a = (2G^T)^{-1a}_b V_+^b, \quad W_-^a = (2G)^a_b V_-^b. \quad (2.8)$$

In terms of these variables, the equations of motion now read

$$\begin{aligned} \partial_+ V_- - \partial_- V_+ - i[V_+, V_-] &= 0, \\ \partial_+ W_- - \partial_- W_+ - i[W_+, W_-] &= 0. \end{aligned} \quad (2.9)$$

These flatness conditions for the vector fields V and W are similar to the zero curvature condition of integrable systems. What is missing is a spectral parameter dependence that will provide by power expansion an infinite number of conserved currents. We will now derive conditions for a zero curvature with a spectral parameter dependence (a Lax pair) to exist. This will be the first integrability condition. The idea is to find a one parameter family of matrices that interpolate between the equations of motion. It is convenient for later use to work with a special linear combination of the V_\pm^a connection:

$$\begin{aligned} M_+^a &\equiv (H_+^{\frac{1}{2}} (2G^T)^{-1})^a_b V_+^b, & H_+ &= 1 - 4GG^T, \\ M_-^a &\equiv (H_-^{\frac{1}{2}})^a_b V_-^b, & H_- &= 1 - 4G^T G. \end{aligned}$$

Define now an interpolating connection as follows:

$$B_\pm^a(x; \lambda) = N_{\pm b}^a(\lambda) M_\pm^b(x), \quad (2.10)$$

¹ We thank Bogdan Morariu for clarifying this point for us.

where $N_{\pm b}^a$ are constants, to be determined as functions of the spectral parameter λ . The vector field B must satisfy the zero curvature condition

$$\partial_+ B_- - \partial_- B_+ - i[B_+, B_-] = 0, \quad (2.11)$$

with boundary conditions

$$\begin{aligned} B_{\pm}^a(x; \lambda = \lambda_0) &= V_{\pm}^a(x), \\ B_{\pm}^a(x; \lambda = \lambda_1) &= W_{\pm}^a(x). \end{aligned} \quad (2.12)$$

In order to proceed we rewrite, after some manipulation, the equations of motion as follows:

$$\begin{aligned} \partial_- M_+^a + A_{-pq}^a M_+^p M_-^q &= 0, \\ \partial_+ M_-^a + A_{+pq}^a M_-^p M_+^q &= 0, \end{aligned} \quad (2.13)$$

where

$$A_{\pm abc} = -(H_{\mp}^{-\frac{1}{2}})_a^r (H_{\mp}^{-\frac{1}{2}})_b^s (H_{\pm}^{-\frac{1}{2}} G_{\mp})_c^t f_{rst} + (H_{\mp}^{-\frac{1}{2}} G_{\pm})_a^r (H_{\mp}^{-\frac{1}{2}} G_{\pm})_b^s (H_{\pm}^{-\frac{1}{2}})_c^t f_{rst}, \quad (2.14)$$

and for convenience we defined $G_+ = 2G^T$ and $G_- = 2G$. Note that $A_{\pm abc}$ is antisymmetric in its first two indices. The zero curvature reads

$$N_{-b}^a \partial_+ M_-^b - N_{+b}^a \partial_- M_+^b + f_{bc}^a N_{+p}^b N_{-q}^c M_+^p M_-^q = 0. \quad (2.15)$$

Using equations (2.13) and equating the coefficients of $M_+^p M_-^q$ to zero we get the first integrability condition

$$N_{+b}^a A_{-pq}^b - N_{-b}^a A_{+qp}^b + f_{bc}^a N_{+p}^b N_{-q}^c = 0 \quad (2.16)$$

This condition gives us a Lax pair and an infinite number of resulting conservation laws (see the section 4). One must also show that these conserved quantities are mutually commuting; that is, their Poisson brackets vanish. This will be the second integrability condition, derived in section 4.

Equations (2.16) are in general an overdetermined algebraic system with $(\dim G)^3$ equations for $2(\dim G)^2$ variables $N_{\pm b}^a$. In fact, since these equations are nonlinear, this counting is misleading. What we have actually is a system of polynomial equations for the variables $N_{\pm b}^a$. The locus of the polynomials defines an algebraic variety \mathcal{M} and the condition of integrability is not that $\mathcal{M} \neq \{0\}$ (this is guaranteed since we always have two solutions when B is equal V or W) but that $\dim \mathcal{M} \geq 1$ in order to have a spectral parameter.

Although the system is overdetermined and there are no parametric solutions for a generic coupling constants G_{ab} , there are special interactions for which the model is integrable. This is the subject of the next section.

3. Solutions of the First Integrability Condition

In this section we will construct some solutions to the integrability condition, proving thus, the existence in those models, of an infinite number of conserved currents. In all examples we will make use of the diagonal ansatz. We assume that G and N_{\pm} are diagonal matrices:

$$\begin{aligned} 2G^a_b &= g_b \delta^a_b, \\ N^a_{\pm b} &= n^{\pm}_b \delta^a_b, \end{aligned}$$

Then $A_{\pm abc} = A_{abc}$,

$$A_{abc} = \frac{g_a g_b - g_c}{(1 - g_a^2)^{\frac{1}{2}} (1 - g_b^2)^{\frac{1}{2}} (1 - g_c^2)^{\frac{1}{2}}} f_{abc}, \quad (3.1)$$

and the integrability condition (2.16) reduces to

$$n_a^+ A_{abc} - n_a^- A_{acb} + f_{abc} n_b^+ n_c^- = 0,$$

or, using (3.1) ,

$$\left(n_a^+ \frac{g_a g_b - g_c}{(1 - g_a^2)^{\frac{1}{2}} (1 - g_b^2)^{\frac{1}{2}} (1 - g_c^2)^{\frac{1}{2}}} + n_a^- \frac{g_a g_c - g_b}{(1 - g_a^2)^{\frac{1}{2}} (1 - g_b^2)^{\frac{1}{2}} (1 - g_c^2)^{\frac{1}{2}}} + n_b^+ n_c^- \right) f_{abc} = 0. \quad (3.2)$$

Note that there is no implied sum here.

Example 1: $2G = g\mathbf{1}$. (The Symmetric Case)

In this case, $N_{\pm} = n^{\pm}\mathbf{1}$, and we have one equation for two variables,

$$n^+ \frac{g^2 - g}{(1 - g^2)^{\frac{3}{2}}} + n^- \frac{g^2 - g}{(1 - g^2)^{\frac{3}{2}}} + n^+ n^- = 0, \quad (3.3)$$

with a one parameter family of solutions.

Example 2: $2G = g_1 \mathbf{1} \otimes g_2 \mathbf{1}$

This just gives two copies of the above equation (3.3) .

Example 3: $SU(2)$ with $U(1)$ symmetry. $2G^a_b = g_b \delta^a_b$ with $g_1 = g_2 \neq g_3$.

This suggests the ansatz, $N^a_{\pm b} = n^{\pm}_b \delta^a_b$ with $n_1^{\pm} = n_2^{\pm} \neq n_3^{\pm}$. Then there are three equations for four variables,

$$\begin{aligned} n_1^+ (g_1^2 - g_3) + n_1^- (g_1 g_3 - g_1) + (1 - g_1^2)(1 - g_3^2)^{\frac{1}{2}} n_1^+ n_3^- &= 0, \\ n_3^+ (g_3 g_1 - g_1) + n_3^- (g_3 g_1 - g_1) + (1 - g_1^2)(1 - g_3^2)^{\frac{1}{2}} n_1^+ n_1^- &= 0, \\ n_1^+ (g_1 g_3 - g_1) + n_1^- (g_1^2 - g_3) + (1 - g_1^2)(1 - g_3^2)^{\frac{1}{2}} n_3^+ n_1^- &= 0. \end{aligned} \quad (3.4)$$

The one parameter family of solutions is given by,

$$\begin{aligned}
n_1^+ &= \left(\frac{g_1}{\lambda(1+g_3)(1-g_1^2)^2} (2(g_3 - g_1^2) + \frac{1}{\lambda}g_1(1-g_3) + g_1(1-g_3)\lambda) \right)^{\frac{1}{2}}, \\
n_1^- &= \left(\frac{g_1\lambda}{(1+g_3)(1-g_1^2)^2} (2(g_3 - g_1^2) + \frac{1}{\lambda}g_1(1-g_3) + g_1(1-g_3)\lambda) \right)^{\frac{1}{2}}, \\
n_3^+ &= \frac{1}{(1-g_3^2)^{\frac{1}{2}}(1-g_1^2)} ((g_3 - g_1^2) + \frac{1}{\lambda}g_1(1-g_3)), \\
n_3^- &= \frac{1}{(1-g_3^2)^{\frac{1}{2}}(1-g_1^2)} ((g_3 - g_1^2) + \lambda g_1(1-g_3)),
\end{aligned} \tag{3.5}$$

where $\lambda = \frac{n_1^-}{n_1^+}$ is the free parameter with range $\frac{1}{g_1} \leq \lambda \leq g_1$. Note that the case where we also have $g_2 \neq g_1$ gives six equations for six variables.

Example 4: Symmetric spaces

Let F be a simple group with a subgroup H . Then the Lie algebra \mathcal{F} can be decomposed into the Lie algebra \mathcal{H} and its orthogonal complement \mathcal{K} , which generates the coset F/H . This coset space is a symmetric space if $[\mathcal{K}, \mathcal{K}] \subset \mathcal{H}$. In what follows we will label the generators of H with greek indices (i.e. λ^α), and the generators of F/H with latin indices (i.e. λ^a), and when we don't want to specify between them, we'll use dotted latin indices (i.e. $\lambda^{\dot{a}}$).

To have a coset (symmetric) space in the present model we choose

$$2G^{\dot{a}}_{\dot{b}} = g_{\dot{b}}\delta^{\dot{a}}_{\dot{b}},$$

with $g_a = g$ and $g_\alpha = 1$. This assures that the currents in \mathcal{H} are set to zero, resulting in a coset model. We cannot use (2.16) because $A_{\pm abc}$ is not defined in this case (H_\pm is not invertible), but instead, we have to use (2.9) and (2.11) :

$$\begin{aligned}
\partial_+ V_-^{\dot{a}} - \partial_- V_+^{\dot{a}} + f_{\dot{b}\dot{c}}^{\dot{a}} V_+^{\dot{b}} V_-^{\dot{c}} &= 0, \\
\partial_+ W_-^{\dot{a}} - \partial_- W_+^{\dot{a}} + f_{\dot{b}\dot{c}}^{\dot{a}} W_+^{\dot{b}} W_-^{\dot{c}} &= 0, \\
\partial_+ B_-^{\dot{a}} - \partial_- B_+^{\dot{a}} + f_{\dot{b}\dot{c}}^{\dot{a}} B_+^{\dot{b}} B_-^{\dot{c}} &= 0.
\end{aligned} \tag{3.6}$$

The choice of G leads to the ansatz:

$$B_\pm^{\dot{a}}(x; \lambda) = r_\pm^{\dot{a}}(\lambda) V_\pm^{\dot{a}}(x),$$

with $r_a^\pm(\lambda) = r^\pm(\lambda)$ and $r_\alpha^\pm(\lambda) = 1$. Then, we can rewrite (3.6) as

$$\begin{aligned} (1 - g_{\dot{a}}^2) \partial_+ V_{-\dot{a}} + f_{\dot{a}\dot{b}\dot{c}} (1 - \frac{g_{\dot{a}} g_{\dot{c}}}{g_{\dot{b}}}) V_{+\dot{b}} V_{-\dot{c}} &= 0, \\ (1 - \frac{1}{g_{\dot{a}}^2}) \partial_- V_{+\dot{a}} + f_{\dot{a}\dot{b}\dot{c}} (1 - \frac{g_{\dot{c}}}{g_{\dot{a}} g_{\dot{b}}}) V_{+\dot{b}} V_{-\dot{c}} &= 0, \\ r_{\dot{a}}^- \partial_+ V_{-\dot{a}} - r_{\dot{a}}^+ \partial_- V_{+\dot{a}} + f_{\dot{a}\dot{b}\dot{c}} r_{\dot{b}}^+ r_{\dot{c}}^- V_{+\dot{b}} V_{-\dot{c}} &= 0. \end{aligned} \quad (3.7)$$

Note that \dot{a} is fixed (no implied sum). Now if \dot{a} is in \mathcal{H} , then the first two equations in (3.6) are the same, and from the third one we get the condition $r^+(\lambda)r^-(\lambda) = 1$. The fact that the first two equations become the same is a sign of gauge invariance. To fix a gauge we choose $V_{-\alpha} = 0$. Then in this gauge, if \dot{a} is in \mathcal{K} , the first two equations in (3.7) solve the third one without further conditions. Hence, for symmetric spaces, we get one equation for two variables

$$r^+(\lambda)r^-(\lambda) = 1. \quad (3.8)$$

4. The Poisson Bracket – Periodic Boundary Condition

Having a Lax pair at hand, we can construct conserved quantities in the time variable x_+ , if we also impose periodicity in the space coordinate x_- . We first define the following quantity:

$$\mathcal{U}(x_+, x_-; \lambda) = P e^{-i \int_{x_-}^{x_+} B_-(x_+, x'_-; \lambda) dx'_-}, \quad (4.1)$$

and take B_\pm periodic in x_- with period 2π . The integral goes between x_- and $x_- + 2\pi \sim x_-$. This quantity satisfies the equation

$$\partial_- \mathcal{U} - i[B_-, \mathcal{U}] = 0, \quad (4.2)$$

which can be taken as the definition ² of path ordering in (4.1). More importantly, the trace of this matrix, $U(\lambda) = \text{Tr } \mathcal{U}(x_+, x_-; \lambda)$, is conserved:

$$\begin{aligned} \partial_+ U &= i \int_{x_-}^{x_+} \text{Tr}(\partial_+ B_- \mathcal{U}) dx'_- \\ (\text{Bianchi id.}) &= i \int_{x_-}^{x_+} \text{Tr}((\partial_- B_+ + i[B_+, B_-]) \mathcal{U}) dx'_- \\ (\text{integration by parts}) &= -i \int_{x_-}^{x_+} \text{Tr}(B_+ (\partial_- \mathcal{U} - i[B_-, \mathcal{U}])) dx'_- \\ (\text{by (4.2)}) &= 0. \end{aligned} \quad (4.3)$$

² This is the same as defining the path ordered exponential with the “upper” limit fixed, i.e., $P e^{\int_a^b dx A(x)} = 1 + \int_a^b dx A(x) + \int_a^b dx \int_x^b dx' A(x) A(x') + \dots$. Note that this is the opposite of the usual definition where the “lower” limit is fixed.

We see that U is a conserved quantity and upon expanding it in powers of λ we get infinite number of conserved currents. Another possible way of increasing the list of conserved quantities is to choose B to live in different representations of the Lie algebra. Since U does not depend on x_+ , we drop its dependence from U as well as the subindex $-$ from x_- . It is convenient to think on the $+$ direction as “time” and the $-$ direction as “space”. The next step is to find the algebra of the conserved currents. In particular if they all commute then we have infinite number of conserved quantities in involution, which is the trademark of integrability. We want to calculate the Poisson brackets of $U(\lambda)$ with $U(\mu)$:

$$\begin{aligned}
\{U(\lambda), U(\mu)\} &= \int dx dy \frac{\delta U(\lambda)}{\delta B_-^a(x; \lambda)} \frac{\delta U(\mu)}{\delta B_-^b(y; \mu)} \{B_-^a(x; \lambda), B_-^b(y; \mu)\} \\
&= - \int dx dy U_a(x; \lambda) U_b(y; \mu) \{B_-^a(x; \lambda), B_-^b(y; \mu)\} \\
&= - \int dx dy U_a(x; \lambda) U_b(y; \mu) N_{-c}^a(\lambda) N_{-d}^b(\mu) \{M_-^c(x), M_-^d(y)\}.
\end{aligned} \tag{4.4}$$

The Poisson brackets for the M 's were calculated in [1] :

$$\begin{aligned}
\{M_-^c(x), M_-^d(y)\} &= - \frac{4\pi}{n} \kappa^{cd} \delta'(x - y) + \frac{4\pi}{n} F_{+k}^{cd} M_-^k(x) \delta(x - y) \\
&\quad + \frac{2\pi}{n} \epsilon(x - y) E_{kl}^{cd}(x, y) M_-^k(x) M_-^l(y),
\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}
E_{kl}^{cd}(x, y) &= A_{+k}^c A_{+l}^d (P e^{\int_x^y A_{-a} M_-^a(x') dx'})_{rs}, \quad (A_{-c})_{ab} = A_{-abc}, \\
F_{+abc} &= (H_-^{-\frac{1}{2}})_a^r (H_-^{-\frac{1}{2}})_b^s (H_-^{-\frac{1}{2}})_c^t f_{rst} - 8 (H_-^{-\frac{1}{2}} G^T)_a^r (H_-^{-\frac{1}{2}} G^T)_b^s (H_-^{-\frac{1}{2}} G^T)_c^t f_{rst}.
\end{aligned} \tag{4.6}$$

Notice that the Poisson brackets include a non local term! It is clear that the only way to get a local algebra for the Poisson brackets for our conserved fields is that the coefficient of $\epsilon(x - y)$ is a derivative such that by integration by parts the derivative acts on ϵ to give a δ function. For a generic coupling this can not happen, but exactly for the models that

satisfy the integrability condition this is true!

$$\begin{aligned}
\{U(\lambda), U(\mu)\}_{non-local} &= \\
&= \frac{2\pi}{n} \int dx dy U_a(x; \lambda) U_b(y; \mu) N_{-c}^a(\lambda) N_{-d}^b(\mu) \epsilon(x-y) E_{kl}^{cd}(x, y) M_{-}^k(x) M_{-}^l(y) \\
&= \frac{2\pi}{n} \int dx U_a(x; \lambda) N_{-c}^a(\lambda) A_{+k}^{c \ r} M_{-}^k(x) \\
&\quad \int dy \epsilon(x-y) U_b(y; \mu) (N_{-d}^b(\mu) A_{+l}^{d \ s}) (P e^{\int_x^y A_{-k} M_{-}^k(x') dx'})_{rs} M_{-}^l(y) \\
&= \frac{2\pi}{n} \int dx U_a(x; \lambda) N_{-c}^a(\lambda) A_{+k}^{c \ r} M_{-}^k(x) \\
&\quad \int dy \epsilon(x-y) U_b(y; \mu) (N_{+d}^b(\mu) A_{-l}^{ds} + f_{tu}^b N_{+}^{ts}(\mu) N_{-l}^u(\mu)) \\
&\quad \times (P e^{\int_x^y A_{-k} M_{-}^k(x') dx'})_{rs} M_{-}^l(y),
\end{aligned} \tag{4.7}$$

where the last step follows from the integrability condition, (2.16). Using the following identities

$$\begin{aligned}
\partial_y U_t(y; \mu) &= -f_{tu}^b B_{-}^u(y; \mu) U_b(y; \mu) = -f_{tu}^b N_{-l}^u(\mu) M_{-}^l(y) U_b(y; \mu), \\
\partial_y (P e^{\int_x^y A_{-k} M_{-}^k(x') dx'})_r^d &= -A_{-l}^{ds} M_{-}^l(y) (P e^{\int_x^y A_{-k} M_{-}^k(x') dx'})_{rs},
\end{aligned} \tag{4.8}$$

we can write now the integrand of the y integral as a total derivative

$$\begin{aligned}
&U_b(y; \mu) (N_{+d}^b(\mu) A_{-l}^{ds} + f_{tu}^b N_{+}^{ts}(\mu) N_{-l}^u(\mu)) (P e^{\int_x^y A_{-k} M_{-}^k(x') dx'})_{rs} M_{-}^l(y) = \\
&- \partial_y \left(U_b(y; \mu) N_{+d}^b(\mu) (P e^{\int_x^y A_{-k} M_{-}^k(x') dx'})_r^d \right).
\end{aligned} \tag{4.9}$$

The contribution of the non-local piece is therefore

$$\begin{aligned}
\{U(\lambda), U(\mu)\}_{non-local} &= \frac{4\pi}{n} \int dx U_a(x; \lambda) U_b(x; \mu) N_{-c}^a(\lambda) N_{-d}^b(\mu) A_{+l}^{c \ d} M_{-}^l(x) \\
&= \frac{2\pi}{n} \int dx U_a(x; \lambda) U_b(x; \mu) (N_{-c}^a(\lambda) N_{+d}^b(\mu) A_{+l}^{c \ d} - N_{+c}^a(\lambda) N_{-d}^b(\mu) A_{+l}^{d \ c}) M_{-}^l(x),
\end{aligned} \tag{4.10}$$

where in the second step we (anti)symmetrized the right hand side so that the change in sign under $\lambda \leftrightarrow \mu$ is manifest. The contributions from the local terms are easily evaluated:

$$\{U(\lambda), U(\mu)\}_{\delta} = -\frac{4\pi}{n} \int dx U_a(x; \lambda) U_b(x; \mu) N_{-c}^a(\lambda) N_{+d}^b(\mu) F_{+l}^{cd} M_{-}^l(x), \tag{4.11}$$

and

$$\begin{aligned}
\{U(\lambda), U(\mu)\}_{\delta'} &= \frac{2\pi}{n} \int dx U_a(x; \lambda) U_b(x; \mu) \\
&\quad (f_{rs}^a N_{-l}^s(\lambda) N_{-c}^r(\lambda) N_{-d}^b(\mu) \kappa^{cd} - f_{rs}^b N_{-l}^s(\mu) N_{-c}^a(\lambda) N_{-d}^r(\mu) \kappa^{cd}) M_{-}^l(x).
\end{aligned} \tag{4.12}$$

Putting everything together we get

$$\{U(\lambda), U(\mu)\} = \int dx U_a(x; \lambda) U_b(x; \mu) J^{ab}_l(\lambda, \mu) M^l_-(x), \quad (4.13)$$

where

$$\begin{aligned} J^{ab}_l(\lambda, \mu) = & N_{-c}^a(\lambda) N_{+d}^b(\mu) A_{+l}^{cd} - N_{+c}^a(\lambda) N_{-d}^b(\mu) A_{+l}^{dc} - 2N_{-c}^a(\lambda) N_{-d}^b(\mu) F_{+l}^{cd} \\ & + f_{rs}^a N_{-l}^s(\lambda) N_{-c}^r(\lambda) N_{-d}^b(\mu) \kappa^{cd} - f_{rs}^b N_{-l}^s(\mu) N_{-c}^a(\lambda) N_{-d}^r(\mu) \kappa^{cd}. \end{aligned} \quad (4.14)$$

The right hand side of equation (4.13) is zero, if the integrand is a total derivative. This motivates us to try the ansatz

$$\partial_x (U_a(x; \lambda) U_b(x; \mu) C^{ab}(\lambda, \mu)) = U_a(x; \lambda) U_b(x; \mu) J^{ab}_l(\lambda, \mu) M^l_-(x). \quad (4.15)$$

The left hand side of this equation can be computed using the identities of (4.8). This result in the second integrability condition

$$J^{ab}_l(\lambda, \mu) = f_{cd}^a N_{-l}^d(\lambda) C^{cb}(\lambda, \mu) + f_{cd}^b N_{-l}^d(\mu) C^{ac}(\lambda, \mu). \quad (4.16)$$

In the diagonal ansatz, C has the form

$$C^{ab}(\lambda, \mu) = C^a(\lambda, \mu) \kappa^{ab}.$$

For models for which a C^{ab} that satisfies this equation can be found, $\{U(\lambda), U(\mu)\} = 0$, and the conserved quantities are in involution. We will now check the models that satisfy the first integrability condition against this second integrability condition. In all examples we have, the result is the same: The second integrability condition is automatically satisfied. However, we do not know whether in general the second condition follows from the first one.

In example 1, the symmetric model, we take $C^a(\lambda, \mu) = C(\lambda, \mu)$, and then we have one equation for one unknown, $C(\lambda, \mu)$. The condition (4.16) is satisfied, and the conserved quantities are commutative. The generalization to example 2 is trivial. In case of example 3, $SU(2)$ with $C^{ab}(\lambda, \mu) = C^a(\lambda, \mu) \kappa^{ab}$ and with $C^1(\lambda, \mu) = C^2(\lambda, \mu) \neq C^3(\lambda, \mu)$, initially we have three equations for two unknowns, but using the fact that $C^a(\lambda, \mu) = -C^a(\mu, \lambda)$ we see that there are only two equations. So, in this case also, equation (4.16) can be solved, and the model is integrable. We have not checked example 4 in detail, but we suspect that the result is the same.

5. The Poisson Bracket – Open Boundary Condition

In this section we will show that the structure of the model is richer with open boundary conditions. We consider the open interval from $-\infty$ to $+\infty$, with fields vanishing at $\pm\infty$. With these conditions, it is possible to construct more integrals of motion than with the periodic one. These integrals do not commute in general and it is the aim of this section to calculate the algebra they generate. A subset of these integrals are the integrals we saw in the periodic case, and they commute, insuring, thus, the integrability of the model.

The calculation of the Poisson brackets is the same as the periodic case, the only difference will come from the boundary. Defining

$$\mathcal{U}_\lambda(x, y) = P e^{-i \int_x^y B_-(x'; \lambda) dx'}, \quad (5.1)$$

then, any matrix element $U^{ab}(\lambda) = \mathcal{U}_\lambda^{ab}(-\infty, \infty)$, and not just the trace, is conserved, provided that $B_+(\pm\infty, \lambda) = 0$. This follows from manipulations similar to (4.3), noticing that the boundary terms coming from $\pm\infty$ can be dropped. Of course, B_- also must vanish sufficiently fast at $\pm\infty$ for the integral to exist. For the sake of simplicity, we will also take B in the adjoint representation.

Define now

$$U_c^{ab}(x; \lambda) = (\mathcal{U}_\lambda(-\infty, x) t_c \mathcal{U}_\lambda(x, \infty))^{ab}, \quad (5.2)$$

then

$$\begin{aligned} \{U^{aa'}(\lambda), U^{bb'}(\mu)\} &= \int dx dy \frac{\delta U^{aa'}(\lambda)}{\delta B_-^c(x; \lambda)} \frac{\delta U^{bb'}(\mu)}{\delta B_-^d(y; \mu)} \{B_-^c(x; \lambda), B_-^d(y; \mu)\} \\ &= - \int dx dy U_c^{aa'}(x; \lambda) U_d^{bb'}(y; \mu) N_{-k}^c(\lambda) N_{-l}^d(\mu) \{M_-^k(x), M_-^l(y)\}. \end{aligned} \quad (5.3)$$

Being careful to take into account the boundary contribution from the non-local term in the Poisson brackets of the M 's, we get

$$\begin{aligned} \{U^{aa'}(\lambda), U^{bb'}(\mu)\} &= \frac{2\pi}{n} \int dx U_c^{aa'}(x; \lambda) U_d^{bb'}(x; \mu) J_e^{cd}(\lambda, \mu) M_-^e(x) \\ &\quad + \frac{2\pi}{n} (U(\lambda) t_c)^{aa'} (t_d U(\mu))^{bb'} U^{lk}(0) N_{-k}^c(\lambda) N_{-l}^d(\mu) \\ &\quad - \frac{2\pi}{n} (t_c U(\lambda))^{aa'} (U(\mu) t_d)^{bb'} U^{kl}(0) N_{-k}^c(\lambda) N_{-l}^d(\mu), \end{aligned} \quad (5.4)$$

where J_c^{ab} is defined in (4.14), and $U^{ab}(0) = U^{ab}(\lambda = \lambda_0)$. If J_c^{ab} also satisfies the second integrability condition (4.16) then the integrand can be written as a derivative. The only contributions come from the boundary and the final result is

$$\begin{aligned} \{U^{aa'}(\lambda), U^{bb'}(\mu)\} = & \frac{2\pi}{n} \left((U(\lambda)t_c)^{aa'} (U(\mu)t_d)^{bb'} C^{cd}(\lambda, \mu) \right. \\ & - (t_c U(\lambda))^{aa'} (t_d U(\mu))^{bb'} C^{cd}(\lambda, \mu) \\ & + (U(\lambda)t_c)^{aa'} (t_d U(\mu))^{bb'} U^{lk}(0) N_{-k}^c(\lambda) N_{-l}^d(\mu) \\ & \left. - (t_c U(\lambda))^{aa'} (U(\mu)t_d)^{bb'} U^{kl}(0) N_{-k}^c(\lambda) N_{-l}^d(\mu) \right). \end{aligned} \quad (5.5)$$

This algebra, in the symmetric case, can be compared with the algebra obtained without bosonizing the fermions [7]. Both the appearance of cubic terms and the more complicated spectral dependence of the “classical” r -matrix $C^{ab}(\lambda, \mu)$ are quantum effects due to the process of bosonization.

Equation (5.5) exhibits a closed algebra for the matrix elements $U^{ab}(\lambda)$; N ’s and C ’s are the corresponding “structure constants” that depend on the particular model under consideration. The algebra is clearly non-linear, with quadratic and cubic terms in U appearing on the right hand side. We have not succeeded in identifying it with any well-known algebra, although it may possibly bear some relation to the W algebras [8]. It is also the generalization of the algebra Lüscher and Pohlmeyer [9] found for the special case of the fundamental representation of $SU(2)$. Finally we would like to comment on the relation of this algebra to the affine algebra of currents found in the principal chiral model [2]. In the symmetric case (Example 1 of section 3), the Thirring model bears a great resemblance to the principal chiral model; for example, the equations of motion (2.9) in this case reduce to

$$\begin{aligned} g\partial_+ V_- + \partial_- V_+ &= 0, \\ \partial_+ V_- - \partial_- V_+ - i[V_+, V_-] &= 0. \end{aligned} \quad (5.6)$$

For $g = 1$, these are identical to the equations of motion of the principal chiral model. Not surprisingly, the Lax pair and the conserved quantities are in one to one correspondence, with obvious modifications in case $g \neq 1$. However, the algebra (5.5) is quite different from the affine algebra found in [2]. This is because in [2], the Poisson brackets of the conserved quantities were derived from the transformations they generate on the field variables, whereas we have used the standard Poisson structure given by the Lagrangian. The two Poisson structures differ in this case, as well as in the principal chiral model [10].

6. Conclusion

In this paper we derived very general conditions (equations (2.16) and (4.16) and (4.14)) that the coupling constant of the Generalized Thirring Model should satisfy to be integrable, in the sense of having an infinite number of conserved quantities. Some solutions were found for special cases. The algebra generated by the conserved currents was calculated for both periodic and open boundary conditions. In the periodic case the currents are in involution while in the open case we found a non linear algebra which has an Abelian subalgebra of conserved currents in involution.

The directions for further research are numerous. One may, for example, add fermions to the bosonized theory and study supersymmetric models. Or, one can try to generalize to a space dependent metric [11]. There is of course the problem of finding new solutions to the overdetermined algebraic systems eqs. (2.16), (4.16) and (4.14). It is important to understand the non linear algebra between the conserved quantities that emerges in the open boundary case.. We would also like to quantize these models and to study the quantum algebraic structure. As a first check we would like to see if the relation between the coupling constants that ensures integrability holds along the renormalization group flow. For the solutions we have studied in this paper, this result follows rather trivially from symmetry considerations. Finally there are the questions of the relation to irrational conformal field theories (see [12] and references therein), and to integrable perturbations of conformal field theories. We hope to address some of these questions in the future.

References

- [1] K.Bardakçi, Nucl.Phys. **B401** (1994) 191.
- [2] J.H.Schwarz, Nucl.Phys. **B447** (1995) 137.
- [3] K.Bardakçi and L.M.Bernardo, Nucl.Phys. **B450** (1995) 695.
- [4] A.M.Polyakov and P.B.Wiegmann, Phys.Lett. **B131** (1983) 121, **B141** (1984) 223.
- [5] D.Karabali, Q.H.Park and H.Schnitzer, Nucl.Phys. **B323** (1989) 572, Phys.Lett. **B205** (1988) 267.
- [6] L.D.Fadeev in “Integrable Systems, Quantum Groups, and Quantum Field Theory”, L.A.Ibort and M.A.Rodríguez (eds.) (1993) 1, Kluwer Academic Publishers.
- [7] H.J.de Vega, H.Eichenherr and J.M.Maillet, Phys.Letters **B132** (1983) 337, Comm.Math.Phys. **92** (1984) 507, Nucl.Phys. **B240** (1984) 377.
- [8] A.B.Zamolodchikov, Theor.Math.Phys. **63** (1985) 1205.
- [9] M.Lüscher and K.Pohlmeyer, Nucl.Phys.**B137** (1978) 46.
- [10] N.Sochen, “Integrable Generalized Principal Chiral Model”, LBL-38034, UCB-PTH-96/04.
- [11] N.Sochen, “Integrable Generalized Sigma Models on a Group Manifold”, in preparation.
- [12] M.B.Halpern, E.Kiritsis, N.A.Obers and K.Clubok, Phys.Rep. **265** (1996) 1, and hep-th/9501144